The second term for two-neighbour bootstrap percolation in two dimensions

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Graph – $[n]^2 \subset \mathbb{Z}^2$ with $n$ ‘large’
Definition

- Graph
- Initial condition \(- A_0 \sim \bigotimes_{x \in [n]^2} \text{Bernoulli}(p)\) with \(p\) ‘small’
Definition

- Graph
- Initial condition
- Bootstrap dynamics – for $t \in \mathbb{N}$

$$A_{t+1} = A_t \cup \{x \in [n]^2, |N_x \cap A_t| \geq 2\}$$
Definition

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- Bootstrap dynamics – for \( t \in \mathbb{N} \)

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$$A_{t+1} = A_t \cup \{ x \in [n]^2, |N_x \cap A_t| \geq 2 \}$$
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- Bootstrap dynamics – for $t \in \mathbb{N}$

\[ A_{t+1} = A_t \cup \{x \in [n]^2, |N_x \cap A_t| \geq 2\} \]
Definition

- Graph
- Initial condition
- Bootstrap dynamics
- Closure – \( [A] = \bigcup_{t \in \mathbb{N}} A_t \)
Definition

- Graph
- Initial condition
- Bootstrap dynamics
- Closure
- Percolation event – \([A] = [n]^2\)
Definition

- Graph
- Initial condition
- Bootstrap dynamics
- Closure
- Percolation event
- Critical probability –

\[ p_c(n) = \inf \{ p \in [0, 1], \mathbb{P}_p(\text{percolation}) \geq 1/2 \} \]
Modelisation of magnetic materials
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• Information/disease/... spreading in a network
• Fun
Previous results
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- [Aizenman-Lebowitz’88]

\[
\frac{c}{\log n} \leq p_c(n) \leq \frac{C}{\log n}
\]
**Previous results**

- [Aizenman-Lebowitz’88] Scaling

**Ideas**

- **Upper bound:** One can infect a square of ‘critical’ size $1/p$ by finding an infection in each row/column successively. It is found at typical distance $\exp(\Theta(1)/p)$ and easily grows indefinitely.

- **Lower bound:** Rectangles process – if there is percolation, there exists an internally filled rectangle of every size.
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03]

\[
\frac{\pi^2 - \epsilon}{18 \log n} \leq p_c(n) \leq \frac{\pi^2 + \epsilon}{18 \log n}
\]
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics

Ideas

- Upper bound: Only ask for an infection in every second row/column and grow in steps of $1/\sqrt{p}$.
- Lower bound: Hierarchies, disjoint occurrence, pod, quantitative optimality of square shapes...
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08]

\[ \frac{\pi^2 - \varepsilon}{18 \log n} \leq p_c(n) \leq \frac{\pi^2}{18 \log n} - \frac{c}{(\log n)^{3/2}} \]
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08] Upper bound for the second term

Idea:
Use the entropy gain from the choice of the lengths of growth steps instead of fixing them as $1/\sqrt{p}$. 
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08] Upper bound for the second term
- [Gravner-Holroyd-Morris’12]

\[
\frac{\pi^2}{18 \log n} - \frac{C (\log \log n)^3}{(\log n)^{3/2}} \leq p_c(n) \leq \frac{\pi^2}{18 \log n} - \frac{c}{(\log n)^{3/2}}
\]
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08] Upper bound for the second term

Idea:
Consider finer hierarchies starting from size $1/\sqrt{p}$. Compensate the large number of hierarchies with the high cost of having many large seeds.
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08] Upper bound for the second term
- [Bringmann-Mahlburg’12] ‘morally’

\[
\frac{\pi^2}{18 \log n} - \frac{C (\log \log n)^{5/2}}{(\log n)^{3/2}} \leq p_c(n) \leq \frac{\pi^2}{18 \log n} - \frac{c}{(\log n)^{3/2}}
\]
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08] Upper bound for the second term
- [Bringmann-Mahlburg’12]
- [Balogh-Bollobás’03] Critical window size is

\[
\frac{(\log \log n)^{O(1)}}{(\log n)^2}
\]
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08] Upper bound for the second term
- [Bringmann-Mahlburg’12]
- [Balogh-Bollobás’03] Window size

Idea:
Apply [Friedgut-Kalai’96].
Previous results

- [Aizenman-Lebowitz’88] Scaling
- [Holroyd’03] Sharp asymptotics
- [Gravner-Holroyd’08] Upper bound for the second term ‘Morally’ the critical window is

$$\frac{\Theta(1)}{(\log n)^2}$$

- [Bringmann-Mahlburg’12]
- [Balogh-Bollobás’03] Window size
Theorem (H, Morris’19)

\[ p_c(n) = \frac{\pi^2}{18 \log n} - \frac{\Theta(1)}{(\log n)^{3/2}} \]
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\[ p_c(n) = \frac{\pi^2}{18 \log n} - \frac{\Theta(1)}{(\log n)^{3/2}} \]

Remark

The upper bound is from GH.
Using non-increasing and non-disjointly occurring events to compensate the number of hierarchies.
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Key lemmas – strong bounds on the probability of gradual growth.
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- Multiple pods allow taking advantage of atypical rectangles featuring in hierarchies.
• Using non-increasing and non-disjointly occurring events to compensate the number of hierarchies.

• Key lemmas – strong bounds on the probability of gradual growth.

• Multiple pods allow taking advantage of atypical rectangles featuring in hierarchies.

• Optimised amount of growth of a rectangle in one step depending on its size. In particular, a swift divergence is needed above the critical size.
- Bounded number of (large) seeds.
- Small pod.
- Short hierarchies.
- Non-small rectangles are not far from squares.
• What is the constant?
• Is the next error term the window size?
• Can similar results be obtained for higher thresholds (in higher dimensions)?
• Can similar results be obtained for other (critical balanced) models?
Thank you.