Perturbations of a large matrix by random matrices

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Main topic of random matrix theory

We consider a $n \times n$ matrix $X_n = (x_{i,j}^{(n)})$ whose entries are random variables.

The main topic of this field is the study of the eigenvalues and eigenvectors of X_n as $n \to \infty$.

The empirical spectral measure

Let us note $\lambda_1, \ldots, \lambda_n$ the eigenvalues of X_n.
The empirical spectral measure of X_n is the probability measure defined by:

$$\mu_{X_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

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For example, in the case of a Hermitian random matrix X_n, for $A \subseteq \mathbb{R}$:

$$\mu_{X_n}(A) = \frac{1}{n} \#\big\{\lambda_i \in A \; ; \; i \in \{1, \dots, n\}\big\}$$

Wigner's Semicircle Law (1958)

If $X_n = (x_{i,j}^{(n)})$ is a $n \times n$ real symmetric random matrix such that

1. $\mathbb{E}(x_{i,j}^{(n)}) = 0$ for $1 \leq i \leq j \leq n$
2. $\mathbb{E}(|x_{i,j}^{(n)}|^2) = 1$ for $1 \leq i < j \leq n$
3. for all $k \in \mathbb{N}$, $\sup_{i,j} \mathbb{E}(|x_{i,j}^{(n)}|^k) = C(k) < \infty$

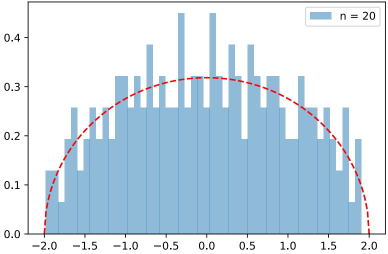
then

$$\mu_{\frac{X_n}{\sqrt{n}}} \xrightarrow[n\to\infty]{\text{dist.}} \mu_{sc}$$

for

$$d\mu_{sc}(t) = \frac{1}{2\pi}\sqrt{4-t^2}\ \mathbb{1}_{[-2,2]}(t)\mathrm{d}t$$

Wigner's Semicircle Law



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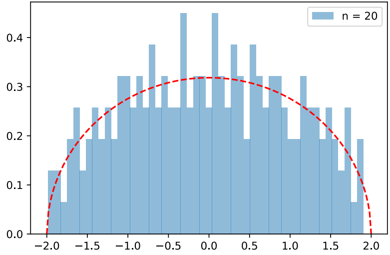

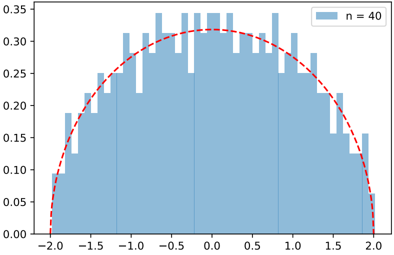

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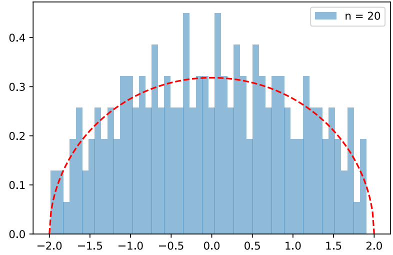

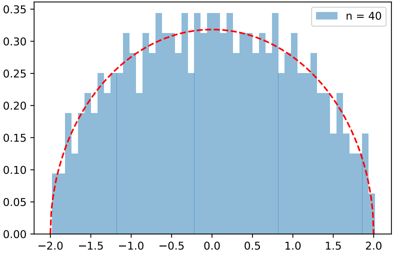

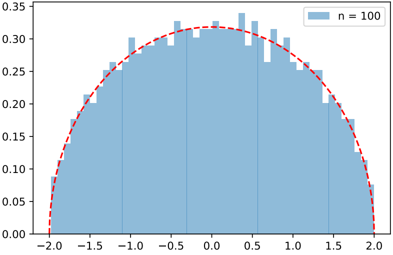

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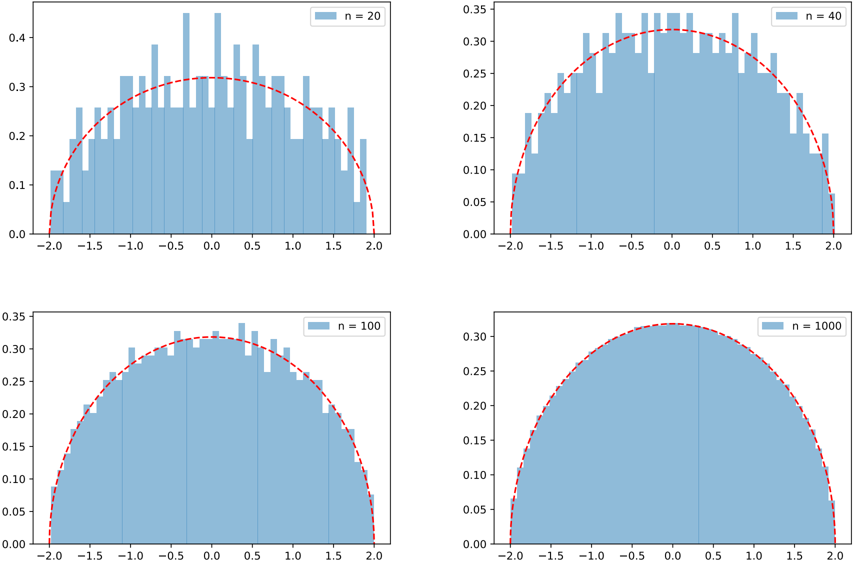

A pertubation problem

How the spectral properties of an operator are altered when the operator is subject to a small perturbation ?

Study of the spectrum of a perturbed matrix

$$H_n$$

- H_n is a deterministic Hermitian matrix.

Study of the spectrum of a perturbed matrix

$$H_n + \quad H_n'$$

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- H_n' is a random Hermitian matrix which operator norm is of order 1.

Study of the spectrum of a perturbed matrix

$$H_n + \varepsilon_n . H'_n$$

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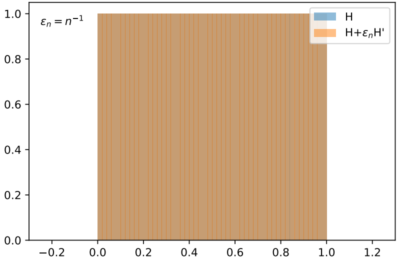
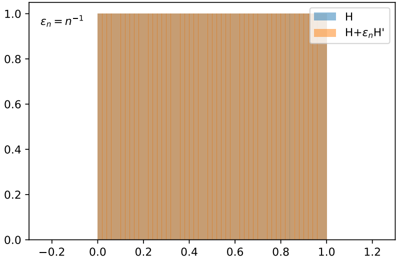
- H'_n is a random Hermitian matrix which operator norm is of order 1.
- (ε_n) is a positive sequence such that $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$



$\varepsilon_n = n^{-1}$
H
H+ε_nH'
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2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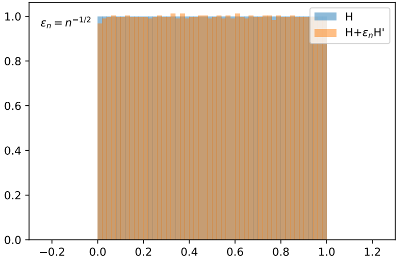
$\varepsilon_n = n^{-1/2}$
H
H+ε_nH'
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
0.2
0.4
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1.0
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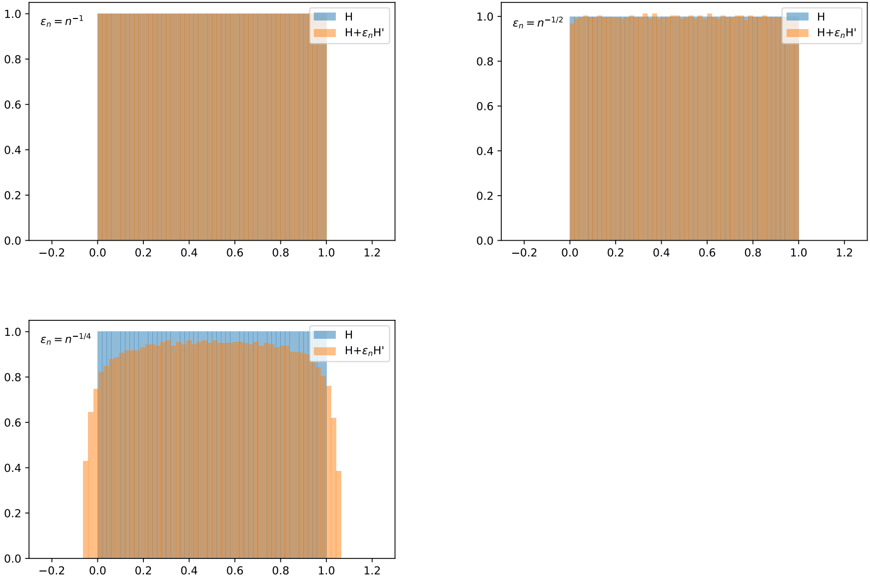
$\varepsilon_n = n^{-1}$
H
H+ε_nH'
$\varepsilon_n = n^{-1/2}$
H
H+ε_nH'
$\varepsilon_n = n^{-1/4}$
H
H+ε_nH'
0.0
0.2
0.4
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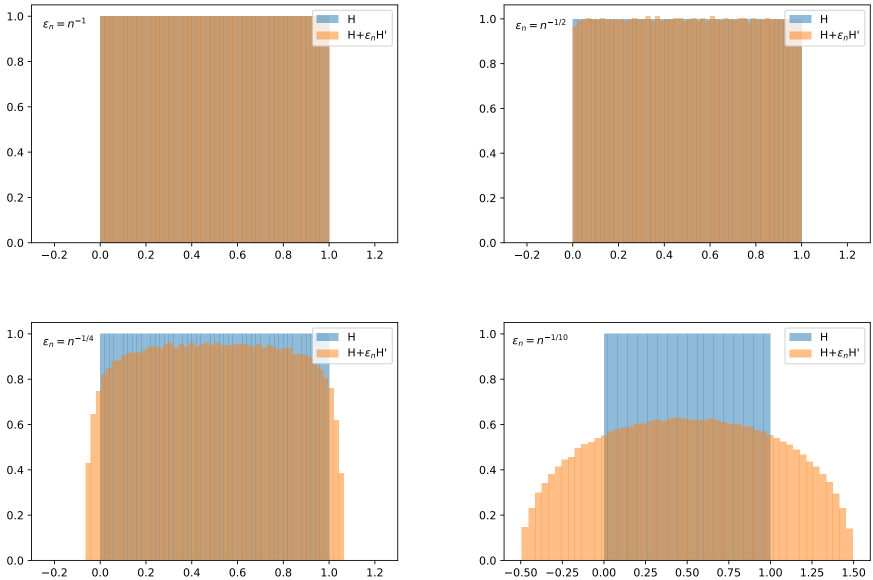
$\varepsilon_n = n^{-1}$
$\varepsilon_n = n^{-1/2}$
$\varepsilon_n = n^{-1/4}$
$\varepsilon_n = n^{-1/10}$
H
H+ε_nH'

Rewriting of the problem

As any Hermitian matrix can be diagonalized by a unitary matrix, U, we can rewrite this problem as :

$$\underbrace{UH_nU^*}_{D_n} + \varepsilon_n . \underbrace{UH'_nU^*}_{X_n}$$

where D_n is a diagonal matrix and X_n an hermitian matrix.

Study of the spectrum of a perturbed matrix

$$D_n^\varepsilon := D_n + \varepsilon_n . X_n$$

Let denotes

- μ_n^ε the empirical spectral distribution of D_n^ε
- μ_n the empirical spectral distribution of D_n

Our aim is to give a pertubative expansion of μ_n^ε around μ_n.

Depending on the order of magnitude of the perturbation, several regimes can appear[1]:

[1] Proved in collaboration with N.Enriquez and F.Benaych-Georges

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- the semi-pertubative regime ($n^{-1} \ll \varepsilon_n \ll 1$):

$$\mu_n^\varepsilon \approx \mu_n + \varepsilon_n^2 \mathrm{d}F$$

Where F is a deterministic function and $\mathrm{d}Z$ a Gaussian random linear form $\mathrm{d}Z$ on $\mathcal{C}^6(\mathbb{R})$, both depends only on the limit parameters of the model.

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Decomposition of the semi-perturbative regime

The case of the semi-pertubative regime

$$\mu_n^{\varepsilon} \;\approx\; \mu_n + \varepsilon_n^2 \mathrm{d}F \qquad\qquad \text{if } n^{-1} \ll \varepsilon_n \ll 1$$

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$$\mu_n^\varepsilon \approx \mu_n + \varepsilon_n^2 \mathrm{d}F + \frac{\varepsilon_n}{n} \mathrm{d}Z \qquad \text{if } n^{-1} \ll \varepsilon_n \ll n^{-1/3}$$

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$$\vdots$$

$$\mu_n^{\varepsilon} \approx \mu_n + (\text{p deterministic terms}) + \frac{\varepsilon_n}{n}\mathrm{d}Z \qquad \text{if } n^{\frac{-1}{2p-1}} \ll \varepsilon_n \ll n^{\frac{-1}{2p+1}}$$

Theorem (F.Benaych-Georges, N.Enriquez and A.M.)

For all compactly supported $\mathcal{C}^6$ function on $\mathbb{R}$, the following convergences hold:

- **Perturbative regime:** *if $\varepsilon_n \ll n^{-1}$, then,*

$$n\varepsilon_n^{-1}(\mu_n^{\varepsilon} - \mu_n)(\phi) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_{\phi}.$$

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- **Critical regime:** *if $\varepsilon_n \sim c/n$, with c constant, then,*

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- **Semi-perturbative regime:** *if $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then,*

$$n\varepsilon_n^{-1}\left((\mu_n^\varepsilon - \mu_n)(\phi) + \varepsilon_n^2\int \phi'(s)F(s)\mathrm{d}s\right) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_\phi .$$

Random term of the expansion

The random term of the expansion is a random field, $(Z_\phi)_{\phi\in\mathcal{C}^6}$, indexed by the space of complex $\mathcal{C}^6$ functions on $\mathbb{R}$, which can be represented as

$$Z_\phi = \int_0^1 \sigma_d(t)\phi'(f(t))\mathrm{d}B_t$$

where, (B_t) is the standard one-dimensional Brownian motion.

$\rightarrow$ σ_d and f are limit parameters of the diagonal entries of X_n and D_n

Idea of the proof

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   In other words, we prove a convergence of the resolvent matrices of D_n^ε and D_n.

2. Then, we extend this convergence to the larger class of compactly supported $\mathcal{C}^6$ functions on $\mathbb{R}$, thanks to the Helffer–Sjöstrand formula and a Lemma of Shcherbina and Tirozzi.

First step of the proof: expansion of the resolvent matrix

For $\varphi_z(x) := \frac{1}{z-x}$ and for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$(\mu_n^\varepsilon - \mu_n)(\varphi_z) \quad = \quad \frac{1}{n} \operatorname{Tr} \frac{1}{z - D_n^\varepsilon} - \frac{1}{n} \operatorname{Tr} \frac{1}{z - D_n}$$

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First step of the proof: expansion of the resolvent matrix

$$
\begin{aligned}
A_n(z) &:= \frac{\varepsilon_n}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} \\
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C_n(z) &:= \frac{\varepsilon_n^3}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} \\
R_n^{\varepsilon}(z) &:= \frac{\varepsilon_n^4}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D^{\varepsilon}}.
\end{aligned}
$$

First step of the proof: expansion of the resolvent matrix

$$A_n(z) \xrightarrow[n\to\infty]{\text{dist.}} Z_{\phi_z}$$

$$B_n(z) := \frac{\varepsilon_n^2}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D}$$

$$C_n(z) := \frac{\varepsilon_n^3}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D}$$

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$$C_n(z) \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

$R_n^{\varepsilon}(z)$ is negligible (in probability)

Second step of the proof: extension to $\mathcal{C}_K^6(\mathbb{R})$ functions

To extend the convergence from functions $\varphi_z(x) = \frac{1}{z-x}$ to $\mathcal{C}_K^6(\mathbb{R})$ functions, we proceed in two steps:

- (Lemma of Shcherbina and Tirozzi) If $s > 5$, then for any $\phi \in \mathcal{H}_s$,

$$n\varepsilon_n^{-1}(\mu_n^\varepsilon(\phi) - \mathbb{E}[\mu_n^\varepsilon(\phi)]) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_\varphi.$$

Lemma

Let $\mathcal{L}_1$ denote the linear span of the functions $\varphi_z(x) := \frac{1}{z-x}$, for $z \in \mathbb{C}\setminus\mathbb{R}$. Then the space $\mathcal{L}_1$ is dense in $\mathcal{H}_s$, for any $s > 0$.

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- (Hellfer-Sjöstrand formula) For any compactly supported function which is $\mathcal{C}^6$ on $\mathbb{R}$, our initial process and $n\varepsilon_n^{-1}(\mu_n^\varepsilon(\phi) - \mathbb{E}[\mu_n^\varepsilon(\phi)])$ are equivalent.

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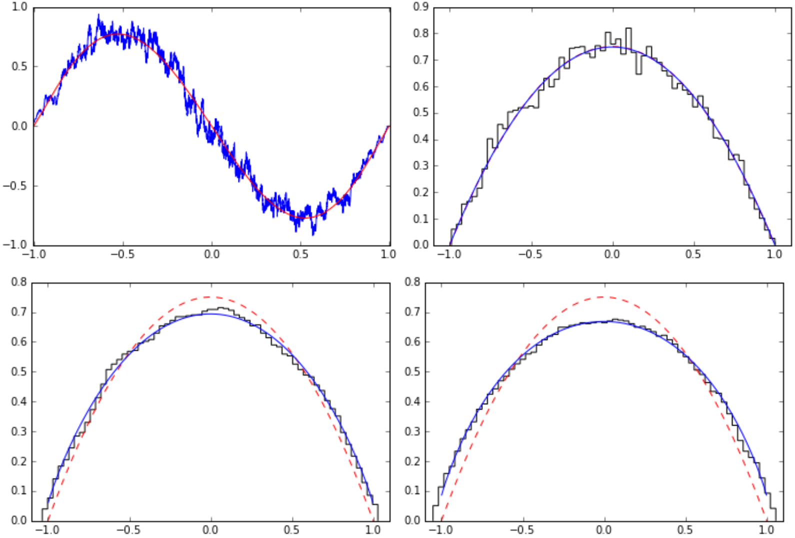
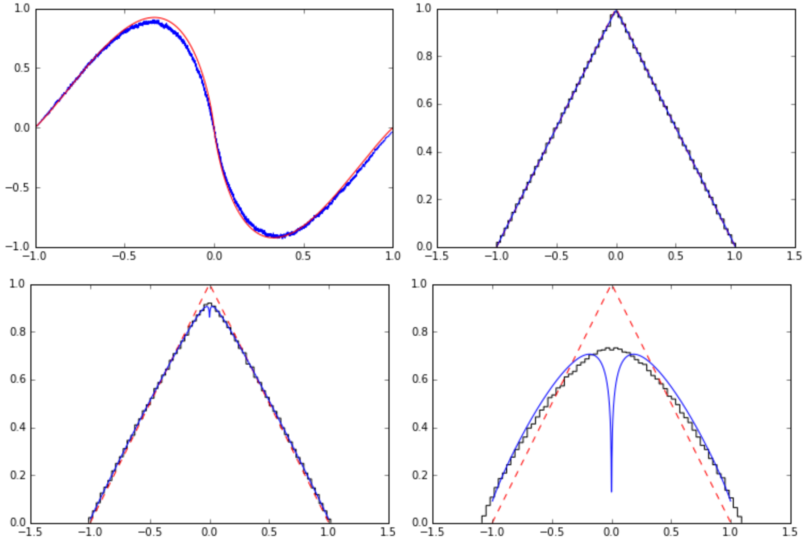
  Thus, as $\mathcal{C}_K^6 \subseteq \mathcal{H}_5$, for any compactly supported function ϕ which is $\mathcal{C}^6$ on $\mathbb{R}$, the processes we studied are also converging to Z_ϕ.

Perturbation of the parabolic pulse distribution by a GOE matrix



Perturbation of the triangular pulse distribution by a GOE matrix



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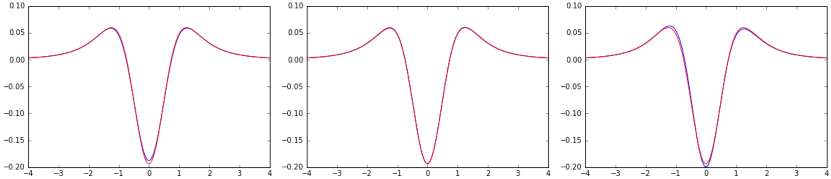

Merci